Aeroelastic Derivatives as a Sensitivity Analysis of Nonlinear Equations

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Nomenclature

$[A]$	= generalized aerodynamic force matrix
$[D]$	= servostructural system damping matrix
Im	= imaginary part of a complex number
$[K]$	= servostructural system stiffness matrix
ℓ	= reference aerodynamic length
$[M]$	= servostructural system mass matrix
$\mathfrak{M}$	= asymptotic Mach number
$\{p\}$	= vector of design parameter
$\{q\}$	= vector of generalized servoaeroelastic coordinates
Re	= Real part of a complex number
s	= $\sigma + j\omega$ = complex circular frequency
V	= asymptotic air speed
V_F	= flutter speed
V_D	= divergence speed
ω_F	= flutter circular frequency
$[W]$	= positively defined weighting matrix
ρ	= asymptotic air density

Introduction

THE use of parametric and optimal design methods that take into account flutter constraints, requires the availability of efficient procedures integrating flutter and sensitivity analysis with respect to design parameters.[1] The way in which these procedures may be integrated in an optimal or parametric design process depends on how the flutter constraint is defined. In many programs it is taken into account by imposing an acceptable flutter speed.[2-4] The drawback of this approch is that there is no warranty about the existence of an acceptable residual damping within the admissible flight envelope[5] and so it can be better to impose the flutter constraint by defining a minimum acceptable damping envelope covering a flight speed range of interest.[6-7]

Depending upon which of these two constraint formulations is chosen, an efficient and practical way to integrate flutter and sensitivity analysis is to formulate the flutter eigenproblem as a nonlinear system of n equations of the form

$$\left(-\omega_F^2[M]+j\omega_F[D]+[K]-\frac{\rho V_F^2}{2}\left[A\left(\mathfrak{M},\frac{\omega_F\ell}{V_F}\right)\right]\right)\{q\}=0$$

$$\tfrac{1}{2}\{q\}^T[W]\{q\}=1 \tag{1}$$

for the direct flutter calculation and

$$\left(s^2[M]+s[D]+[K]-\frac{\rho V^2}{2}\left[A\left(\mathfrak{M},\frac{s\ell}{V}\right)\right]\right)\{q\}=0$$

$$\tfrac{1}{2}\{q\}^T[W]\{q\}=1 \tag{2}$$

for the determination of an aeroelastic eigensolution at any flight condition.[8]

The direct differentiation of Eqs. (1) and (2), with respect to a design parameter p_i, allows us to write the following two systems of linear equations:

$$\left(-\omega_F^2[M]+j\omega_F[D]+[K]-\frac{\rho V_F^2}{2}[A]\right)\frac{\mathrm{d}\{q\}}{\mathrm{d}p_i}$$
$$+\left(-2\omega_F[M]+j[D]-\frac{\rho V_F^2}{2}\left(\frac{\partial[A]}{\partial\omega}\right)_{\omega=\omega_F}\right)\{q\}\frac{\mathrm{d}w_F}{\mathrm{d}pi}$$
$$+\left(-\rho V_F[A]-\frac{\rho V_F^2}{2}\left(\frac{\partial[A]}{\partial V}\right)_{V=V_F}\right)\{q\}\frac{\mathrm{d}V_F}{\mathrm{d}p_i}$$
$$=-\left(-\omega_F^2\frac{\mathrm{d}[M]}{\mathrm{d}p_i}+j\omega_F\frac{\mathrm{d}[D]}{\mathrm{d}p_i}+\frac{\mathrm{d}[K]}{\mathrm{d}p_i}\right)\{q\}$$
$$\{q\}^T[W]\frac{\mathrm{d}\{q\}}{\mathrm{d}p_i}=0 \tag{3}$$

$$\left(s^2[M]+s[D]+[K]-\frac{\rho V^2}{2}[A]\right)\frac{\mathrm{d}\{q\}}{\mathrm{d}p_i}$$
$$+\left(2s[M]+[D]-\frac{\rho V^2}{2}\frac{\partial[A]}{\partial s}\right)\{q\}\frac{\mathrm{d}s}{\mathrm{d}p_i}$$
$$=-\left(s^2\frac{\mathrm{d}[M]}{\mathrm{d}p_i}+s\frac{\mathrm{d}[D]}{\mathrm{d}p_i}+\frac{\mathrm{d}[K]}{\mathrm{d}p_i}\right)\{q\}$$
$$\{q\}^T[W]\frac{\mathrm{d}\{q\}}{\mathrm{d}p_i}=0 \tag{4}$$

that give the derivatives of the desired aeroelastic eigensolutions with respect to any design parameter.[9] Equations (3) and (4) are generally solved by a forward-backward substitution with the LU factorization already available from the solution of the corresponding analysis problem. Since the factorized form of the coefficient matrix always remains the same, irrespective of the design parameter, any calculation of an eigensensitivity requires $\mathcal{O}(n^2)$ operations.

It will be shown that by using an appropriate adjoint system of linear equations, it is possible to reduce the operations required for the calculation of the aeroelastic eigensensitivities to $\mathcal{O}(n)$ with a consequent substantial gain of efficiency, especially when a large number of parameters is involved in the design process.

The use of an adjoint problem for improving the efficiency of the calculation of sensitivities is well known,[10,11] however, its application to the aeroelastic eigenproblems (that will be shown in this Note) is a new improvement to Ref. 9 which first presented the integrated approach of Eq. (1-4).

Sensitivities of Nonlinear Equations

Equations (1-4) can be seen as a particular application of the more general problem of computing the sensitivities of the solution of a set of linear or nonlinear equations whose coefficients depend upon a set of parameters. In fact calling $\{p\}$ a vector of parameters and $\{x\}$ the unknown vector, the derivatives of the unknowns of the following system of nonlinear equations

$$\{F(\{x\},\{p\})\}=0 \tag{5}$$

with respect to p_i, is obtained by solving the following system of linear equations:

$$\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\frac{\mathrm{d}\{x\}}{\mathrm{d}p_i}=-\frac{\partial\{F\}}{\partial p_i} \tag{6}$$

where $[\partial\{F\}/\partial\{x\}]$ is the Jacobian matrix of $\{F\}$.

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If we are interested only in the ℓth component of $d\{x\}/dp_i$ i.e., dx_ℓ/dp_i, we can define a vector $\{I_\ell\}$ whose components are null except for the ℓth element which is equal to one. In this way we obtain

$$\frac{dx_\ell}{dp_i}=\{I_\ell\}^T\frac{d\{x\}}{dp_i}=-\{I_\ell\}^T\left[\frac{\partial\{F\}}{\partial\{x\}}\right]^{-1}\frac{\partial\{F\}}{\partial p_i} \tag{7}$$

In the case of aeroelastic constraints the derivatives are required for only a few components of $\{x\}$, i.e., dV_F/dp_i, ds/dp_i, with respect to a large number of design parameters. Therefore it is better to solve, once and for all, the following adjoint system of equations:

$$\left[\frac{\partial\{F\}}{\partial\{x\}}\right]^T\{\lambda\}=\{I_\ell\} \tag{8}$$

where $\{\lambda\}$ is the adjoint vector. From Eq. (7) we can then write

$$\frac{dx_\ell}{dp_i}=-\{\lambda\}^T\frac{\partial\{F\}}{\partial p_i} \tag{9}$$

The solution of Eq. (8) can be obtained by using the transpose of the LU factorization of the Jacobian matrix, which is already available from the converged iterative solution of Eq. (5), if a Newton-like method is employed. Equation (9) then reduces the number of operations required to calculate dx_ℓ/dp_i to $\mathcal{O}(n)$ compared to the $\mathcal{O}(n^2)$ required by Eq. (6), thus allowing a substantial gain in computational efficiency, especially when the calculation of $\partial\{F\}/\partial p_i$ is also of $\mathcal{O}(n)$ or less as in the aeroelastic case.

We note that Eqs. (8) and (9) are equivalent to the so-called "dummy-load" method used to calculate the sensitivities of structural displacements in the FEM static structural analysis[11,12] with the only difference being that the Jacobian is not necessarily symmetric and sparse.

The reasoning can be trivially extended to second derivatives which can be obtained, after the differentiation of Eq. (6) with respect to p_j, by solving the following linear system of equations:

$$\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\frac{d\{x\}}{dp_i dp_j}=-\left(\frac{\partial}{\partial\{x\}}\left(\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\right)\frac{d\{x\}}{dp_j}+\frac{\partial}{\partial p_j}\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\right)\frac{d\{x\}}{dp_i}-\frac{\partial}{\partial p_j}\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\frac{d\{x\}}{dp_j}-\frac{\partial^2\{F\}}{\partial p_i\partial p_j} \tag{10}$$

which maintain the same coefficient matrix as that of Eq. (6).

Then $dx_\ell/dp_i dp_j$ is given by

$$\frac{dx_\ell}{dp_i dp_j}=\{\lambda\}^T\left\{-\left(\frac{\partial}{\partial\{x\}}\left(\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\right)\frac{d\{x\}}{dp_j}+\frac{\partial}{\partial p_j}\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\right)\frac{d\{x\}}{dp_i}-\frac{\partial}{\partial p_i}\left[\frac{\partial\{F\}}{\partial\{x\}}\right]\frac{d\{x\}}{dp_j}-\frac{\partial^2\{F\}}{\partial p_i\partial p_j}\right\} \tag{11}$$

To calculate the second derivatives, it is first necessary to calculate all the first-order derivatives $d\{x\}/d\{p\}$ with respect to the set of parameters of interest. The operations needed to evaluate the right-hand side of Eq. (10) can be so substantial that the use of Eq. (11) cannot change the gain from $\mathcal{O}(n^2)$ to $\mathcal{O}(n)$. Nevertheless, the use of Eqs. (9) and (10) is still recommended since the calculation of $\mathcal{O}(n^2)$ operations, needed to solve Eqs. (6) and (10) from the LU factorization of $[\partial\{F\}/\partial\{x\}]$, must be performed only once for each ℓth component.

Specialization of Aeroelastic Eigenproblems

In applying Eqs. (8) and (9) to Eqs. (3) and (4), we must note that in Eq. (1), ω_F and V_F are two real independent eigenvalues. Thus, while Eq. (1) is a system of $2(n+1)$ real equations with $\{x\}$ real and defined by Eq. (9) as

$$\{x\}=\begin{Bmatrix}\mathrm{Re}\{q\}\\ Im\{q\}\\ \omega_F\\ V_F\end{Bmatrix} \tag{12}$$

Eq. (2) is a system of $(n+1)$ complex equations with $\{x\}$ defined by

$$\{x\}=\begin{Bmatrix}\{q\}\\ s\end{Bmatrix} \tag{13}$$

If the flutter constraint is then expressed as a minimum acceptable speed, the corresponding adjoint vector of Eq. (8) is obtained by using the real $\{I_{2(n+1)}\}$ vector. If it is expressed by a minimum damping envelope, either in terms of σ or of relative damping, the adjoint eigenvector is derived from a complex $\{I_{n+1}\}$ vector. To constrain the flutter frequency, the corresponding adjoint vector can be calculated by using a real $\{I_{2n+1}\}$ vector in Eq. 6.

It is also worth noting that this technique can be extended to the divergence eigenproblem[9] which is simply Eq. (1) with $[M]=[D]=0$, $[A]$ real, i.e., $\omega=0$, which gives a real $\{x\}$ defined by

$$\{x\}=\begin{Bmatrix}\{q\}\\ V_D\end{Bmatrix} \tag{14}$$

and which requires $\{I_{n+1}\}$ to calculate dV_D/dp_i.

This approach can be useful if the divergence constraint is expressed by a minimum acceptable divergence speed for assigned altitudes or Mach numbers. In the case in which it is given in terms of dynamic pressures against Mach numbers, it can be more conveniently solved by standard eigenproblem analysis and sensitivity methods.[10]

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